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PRACTICAL INFORMATION

Major news:

- last home assignment returned today (in the break),
- (official) course evaluations at approx 10:45 today,
- course syllabus posted to webpage (already last week),
- you need to notify me about **your choice on midterm** (by Wednesday, Dec 6th).

Today's lecture:

- Exam: **MONDAY 11/12, 9AM-12PM, AVC 287N**,
 - * exam practical remarks,
 - * exam questions (types, calculations),
- **review slides**: new or partly new slides, 13L-4/.../14,
- also a **list of review slides** from previous lectures (appendix, 13L-15),
- **review of exam questions** from final exam 2015:
 1. descriptive statistics/two-sample inference,
 2. proportion data/sample size,
 3. ANOVA/regression.

EXAM PRACTICAL REMARKS

Start time and number of questions depend on your midterm choice:

- use midterm: start at 10am, 2 questions — 35%,
- drop midterm: start at 9am, 3 questions — 50%.

Exam rules: same as for midterm (see instructions at webpage), in particular:

- open book (all aids are allowed), except a computer-like device, although cell phones and tablets can be used (in flight mode):
 - * as a calculator (with basic calculation functions, no statistical software),
 - * to access course notes and electronic textbook material.

Some hints and advices: (to use or not...)

- layout — essential requirements: readability, and a clear distinction between what is *in* your answer and what is not — **don't** write first a draft and then a final version,
- conclusions should be part of all analyses,
- statistical model should be part of all data analysis,
- explicit calculations may prevent loss of points due to typing errors (or the like),
- errors: if you realize an error and do not have time to correct it: write what is wrong, what should have been done and how the error would affect the result.

EXAM QUESTIONS

2 or 3 major questions, with subquestions — possible types:

- choice of statistical model and analysis:
 - * carry out analysis when calculations manageable (see below),
 - * or base analysis on Minitab print + extra calculations,
 - * or outline analysis if calculations not manageable,
- probability calculations manageable (see below),
- multiple choice (one or several correct answers),
- brief essay (10 lines) to explain statistical concept, to interpret result, or to explain why something is wrong.

Calculations by complexity — some examples (not all cases covered)

Possible calculations	Too complex calculations
simple probabilities (e.g., $1 - p$, $(1 - p)^n$, simple binomial)	statistical analysis of normal distrib. models without calculation help (given statistics or computer output)
standardization in normal distribution	probability plots, residual plots
t -test and CI (given suitable estimates)	prediction intervals in regression
backtransformation of means to original scale	rank-based nonparametric tests
proportions (with CI)	χ^2 -tests
sample size for precision	power calculations

TYPICAL FORMULAS FOR EXAM

List of formulas (non-exhaustive!) of potential relevance for exam:

- 1L: suspected outlier rule,
- 3L: probability rules (e.g.: addition, multiplication); means and variances for random variables,
- 4L: normal distrib.: calc. of probabilities + percentiles (incl. standardization, z -score); binomial distrib.: calc. of simple probabilities, mean, stand.dev.,
- 5L: standard error (for mean); z -test and CI for 1-sample normal,
- 6L: t -test and CI for 1-sample and 2-sample normal,
- 7L: CI and test for 1 and 2 proportion(s); sign test,
- 8L: expected value (cell) for X^2 -test,
- 9L: one-way ANOVA: equal variance guideline; LSD-value; ANOVA table; t -test and CI for mean/difference between means,
- 10L: t -test and CI for regression parameters; ANOVA table; prediction (without intervals); t -test for correlation,
- 11L: two-way ANOVA: similar to 9L,
- 12L: sample size for precision (1-sample normal/proportion, detection of disease).

OUTLINE OF STATISTICAL ANALYSIS

Data description:

- descriptive statistics: plots, tables, simple statistics — for the purposes:
 - * overview of the data,
 - * detect errors / “different” observations (**outliers**),
 - * focus attention on what’s relevant.

Statistical model: we formulate statistical models containing theoretical distributions and unknown parameters, in order to

- make clear the assumptions (and utilize them),
 - let (statistical) parameters¹ represent issues of interest,
- “All models are approximations, but some are useful” (George Box).

Estimation: our information about the parameters from the observed data is summarized in **statistics** or **estimates**:

- quantities calculated from the data to give **possible parameter values**,
- aim of statistical methodology: obtain **estimates as close to true values** as possible²,
- all estimates should be accompanied by a **measure of uncertainty**, such as **standard error** or **confidence interval**.

¹ In our statistical setup, parameters are fixed, unknown numbers describing a population.

² In most cases, it is unlikely and implausible that an estimate would be exactly equal to the true value.

Model check:

- comparison of observed distribution and assumed theoretical distribution (using estimated parameters),
- **methods**: graphical (plots) or numerical (tests),
- if model unsatisfactory, **start over with new model**³.

Hypothesis testing:

- formulate, using model parameters, **null hypothesis** H_0 (model simplification) and **alternative hypothesis** H_a ,
- the **test statistic** and associated ***P*-value** summarize our evidence **against null hypothesis**, which we may **reject** (low *P*) or **not reject** (high *P*).

Final Model:

- **simplest possible** after model reduction(s),
- if necessary, re-estimate model parameters (with CI).

Conclusion / Presentation:

- summary of test results,
- illustrations of **implications** of the final model, e.g. prediction; also, **presentation** of estimates (with SE or CI).

³ For less serious violations of model assumptions, it may alternatively be reasonable to **proceed with caution**.

GRAPHS IN STATISTICAL ANALYSIS

Graphs have different **purposes**:

- **Descriptive**: show the shape of the distribution in a dataset,
 - * dotplot and stemplot (raw data),
 - * histogram (grouped raw data),
 - * boxplot (schematic for descriptive statistics),
 - * scatterplot (raw data, 2 variables),

note: small datasets are best illustrated by raw data plots, and more data \Rightarrow more schematic/grouped plots,
- **Model check**: graphical assessment of one or more model assumptions,
 - * normal probability plots (check normal distribution within groups/samples),
 - * residual plots (check different assumptions in normal distribution models),
- **Presentation**: graphical display of estimates (possibly with measure of uncertainty) from (complex) analysis,
 - * group mean plots with error bars, based on model standard deviation ($\sqrt{\text{MSE}}$), (ANOVA models),
 - * fitted line plots (regression).
 - * interaction plots (2-way ANOVA).

ABOUT STATISTICAL MODELS

What is a statistical model?

- a formal statement/description of the assumptions made for specific statistical inference, such as statistical hypothesis test and confidence (or prediction) interval,
- the assumptions quantify the random variability in the data relative to model parameters (= the constants representing the underlying population).

How to state/write a statistical model? (for the exam) — choose between:

- formal notation with equations and explicit parameters, e.g.
 - i) 1-sample: X_i 's are i.i.d. and $\sim N(\mu, \sigma)$,
 - ii) regression: $Y_i = \beta_0 + \beta_1 \cdot x_i + \varepsilon_i$, with $\varepsilon_i \sim N(0, \sigma)$,
- or a descriptive (but detailed) statement, e.g.
 - i) sample from a normal distribution with unknown mean and standard deviation,
 - ii) a linear relationship: $Y = \beta_0 + \beta_1 \cdot x + \text{error}$,
where the errors are normally distributed with mean 0 and the same standard deviation,
- or any mix (or variation) of these, where assumptions are explicitly stated or clearly understood.

CHOICE OF STATISTICAL MODEL

Some useful questions to ask about the data:

- purpose of study?
- what is the observational unit/experimental unit/subject?
- response/outcome⁴ or explanatory/predictor variable?
- quantitative (including continuous) or categorical (including binary) variable?
- which variables/groupings/classifications should enter into the model? — e.g.,
 - * a single sample (normal, binomial),
 - * two independent samples (normal, binomial, multinomial),
 - * several independent samples (one-way ANOVA, two-way table of counts),
 - * paired observations \Rightarrow single sample for differences,
 - * two-way classification (two-way table of counts, two-way ANOVA),
- quantitative (possibly continuous) variable (explanatory or response) to be used for prediction of another variable? (regression),
- two quantitative (possibly continuous) response variables with no wish to predict one from the other? (correlation),
- transformation? (to achieve normal distribution, homogeneity of variance, linear relation).

⁴ What defines a response variable is that it has (real) random variation.

MULTI-PURPOSE STATISTICAL TESTS

Some statistical tests are **specific** to a single situation/use, e.g. tests for normality and rank-based tests. However, tests named after a probability distribution usually have **multiple** uses.⁵

2 independent samples

Name	Use(s) in course	Instances/Versions
<i>z</i> -test	normal distrib. inference with known σ binomial distrib. inference	1-sample <i>z</i> -test 2-sample [#] <i>z</i> -test (barely covered) <i>z</i> -test for 1 and 2 proportions
<i>t</i> -test (“Student” <i>t</i>)	normal distrib. inference with unknown σ	1-sample and 2-sample [#] <i>t</i> -tests <i>t</i> -test for regression parameters <i>t</i> -test for correlation coefficient <i>t</i> -test for (contrasts and) pairwise comparisons
χ^2 -test (chi-square)	inference for counts rank-based tests	2-way table tests for homogeneity and independence Kruskal-Wallis test
<i>F</i> -test	effects in normal distrib. models	linear regression (slope) factorial effects (ANOVA)

Relations between tests:

- *t*-test (df) with very large df \approx *z*-test,
- *z*-test squared $\sim \chi^2(\text{df}=1)$ -test,
- *t*-test (df) squared $\sim F(\text{df}_1 = 1, \text{df}_2 = \text{df})$ -test.

⁵ The table shows examples from VHM 801 only, many more exist for other statistical models/procedures!

REVIEW: *P*-VALUES

Statistical significance, as determined from *P*-value⁶:

Significant			Non-significant	<i>P</i> -value
***	**	*	NS	
0.1%	1%	5%		

Interpretation and suggested wording (personal preferences, not banning significance):

- *P*-values above 0.05 are usually denoted **non-significant** and interpreted as indicating that the result (observed value of the test statistic) could have occurred by coincidence if H_0 is true, so **that H_0 cannot be rejected**,
- *P*-values just around 0.05 (no matter if just above or below) are **close to significant**, and give **indication** that H_0 may not be true,
- formally, a *P*-value below 0.05 is **evidence against H_0** ,
- *P*-values at 0.01 or below are **clearly significant** and give clear indication that H_0 is false, so that **H_0 should probably be rejected**,
- *P*-values at 0.001 or below are **strongly significant** and show that the data are incompatible with H_0 , so that **H_0 should be rejected**, unless other explanations exist.

⁶ **Warning !** — do not confuse *P*-values and values for *p*'s:

- *P*-values are values (certain probabilities) computed to interpret test results,
- *p*'s are (probability) parameters in particular models, e.g. $B(n, p)$.

REVIEW: ANOTHER LOOK AT t AND t^*

From a (hypothetical) confused individual: **what are all those different t 's:**

t^* , $t(\text{df})$, $t_{1-\alpha}(\text{df})$, t_{obs} , t , \dots

For **confidence intervals**, e.g. with **confidence level 95%** ($1-\alpha$) and **error level 5%** (α), we need in the 1-sample formula:

$$\mu : \bar{X} \pm t^* s / \sqrt{n}, \quad t^* = t_{1-\alpha/2}(\text{df}),$$

a number, $t^* = t_{1-\alpha/2}(\text{df})$, from a t distribution with df degrees of freedom:

- determines the middle 95% of the t distrib., between 2.5% and 97.5% ($\alpha/2$ and $1-\alpha/2$),
- equals the 97.5% percentile in the t distribution,
- can be found in a statistical table (under conf. level 95% or tail area probability 2.5%),
- **Minitab**: Probability Distribution Plot with Right Tail Probability = 0.025.

For a **test** of $H_0: \mu = \mu_0$ (where μ_0 is a known value), we compute the observed value of our t -statistic from the 1-sample formula:

$$t_{\text{obs}} = (\bar{X} - \mu_0) / (s / \sqrt{n}),$$

and the P -value is calculated **from** $P(t(\text{df}) > |t_{\text{obs}}|)$,⁷

- it is a tail area probability, and can be evaluated as $<$ or $>$ specific probabilities in a table, based on the table values below or above $|t_{\text{obs}}|$,
- **Minitab**: Probability Distribution Plot with Right Tail X Value = $|t_{\text{obs}}|$.

⁷ Generally, $t(\text{df})$ refers to the t distribution with df degrees of freedom, sometimes just t when the df are understood.

REVIEW: DEGREES OF FREEDOM

What are those “degrees of freedom” (DF or df) really?

- values to distinguish between different distributions of the same type (distribution types: t , χ^2 , F),
- always positive integer numbers: $1, 2, 3, \dots$,
- in normal models, a large DF for the estimate of σ corresponds to good precision (many observations),

- part of ANOVA tables (e.g., use DFE for t^*),
- specific formulas exist for standard situations:⁸

Model	Data	Purpose	DF
single sample	X_1, \dots, X_n	conf. int. t -test	$n - 1$
two samples (same σ)	X_1, \dots, X_{n_1} Y_1, \dots, Y_{n_2}	conf. int. t -test	$n_1 + n_2 - 2$
1-way ANOVA	$X_{ij}, i = 1, \dots, I$ $j = 1, \dots, n_i$	conf. int. t/F -test	error: $\sum_i n_i - I$ groups: $I - 1$
2-way ANOVA	$X_{ijk}, i = 1, \dots, I$ $j = 1, \dots, J$ $k = 1, \dots, n_{ij}$	conf. int. t/F -test	error: $\sum_{ij} n_{ij} - IJ$ groups: $I - 1$ & $J - 1$ interac: $(I - 1)(J - 1)$
simple linear regression	(Y_i, x_i) $i = 1, \dots, n$	conf. int. t -test	$n - 2$
2-way table	$N_{ij}, i = 1, \dots, I$ $j = 1, \dots, J$	chi-square test	$(I - 1)(J - 1)$

⁸ (technical) All DFs can be interpreted as the difference in number of (free) parameters between two models: a comprehensive model and a reduced model.

REVIEW: GENERAL PRINCIPLES OF TESTS AND CIs

- **Data** X_1, \dots, X_n ,
- **Statistical model** containing a parameter μ , typically a mean parameter.
- **Estimate** $\hat{\mu}$ for μ , based on X_1, \dots, X_n .
- **Standard error** $\text{SE}(\hat{\mu})$, either
 - * estimated from the data, or
 - * known value (rarely realistic in practice),

note: in normal models (with error standard deviation σ) we often have

$$\text{Var}(\hat{\mu}) = A \sigma^2 \quad \text{and} \quad \text{SE}(\hat{\mu}) = \sqrt{A} \sigma,$$

where A is a constant determined by the form of $\hat{\mu}$,

- **Reference distribution** of $\frac{\hat{\mu} - \mu}{\text{SE}(\hat{\mu})} \sim$ percentiles t_p ; **note:** in normal models with estimated $\text{SE}(\hat{\mu})$ the reference distribution is usually a t -distribution,
- **Confidence interval** $(1 - \alpha)$ for μ : $\hat{\mu} \pm t^* \text{SE}(\hat{\mu})$, $(t^* = t_{1-\alpha/2})$
- **Test** of $H_0: \mu = \mu_0$ against $H_a: \mu \neq \mu_0$,

$$\text{test statistic: } t = \frac{\hat{\mu} - \mu_0}{\text{SE}(\hat{\mu})}, \quad \text{P-value: } P = 2 \times \text{P}(t \geq |t_{\text{obs}}|),$$

where $t \sim$ the reference distribution.

APPENDIX: LIST OF POTENTIAL REVIEW SLIDES FOR COURSE

Selected slides on statistical analysis across all sessions (Summary notes may also be useful):

- completely randomized design and block design (2L–9/10),
- binomial setting (4L–10),
- one- or two-sided (5L–18),
- testing by confidence interval (5L–19),
- PLS four-step processes for confidence intervals and tests (5L–21),
- how to find P -values and percentiles (6L–7),
- inference for non-normal data (6L–9),
- inference for proportions — overview/single/two (7L–2/3/4/6),
- nonparametric (distribution-free) methods (7L–8/11),
- sign test (7L–9),
- two-way tables: models, estimation and hypotheses (8L–8/9),
- after the ANOVA table: LSD & Bonferroni (9L–13/14),
- use of residuals for model check (10L–11/12),
- correlation vs. regression (10L–15),
- overview 1-way & 2-way ANOVA (11L–20),
- statistical testing issues (12L–3),
- sample size based on estimation accuracy/power (12L–10/11/13).