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Practical Information

Scheduling and organization:

- first home assignment: will be posted next Thursday (on webpage and Moodle),
- final exam (tentatively) set for Monday, December 13, 9am-12pm
  — any objections?
- we also need to set the midterm exam around October 28th,
  * duration 1 hour,
  * exam is *optional*, but recommended to take it,
- note: summary worksheet postings at Moodle also include answers.

Today's lecture:

- the normal distribution,[1]
- the binomial distribution[2] and Poisson distribution,[3]
- finally arrived at the real thing: statistical inference,[4]
  * parameters and statistics,
  * the first component of inference: parameter estimation.

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[1] PSLS 4e: Chapter 11; IPS7e: Section 1.3; S: Chapter 6.
[2] PSLS 4e: Chapter 12; IPS7e: Section 5.2; S: Chapter 5.
[3] Not a core course topic; PSLS 4e: Chapter 12, IPS 9e: Section 5.3.
[4] PSLS 4e: Chapter 13 (part); IPS7e: Section 5.1 (part); S: Chapter 1 (part).

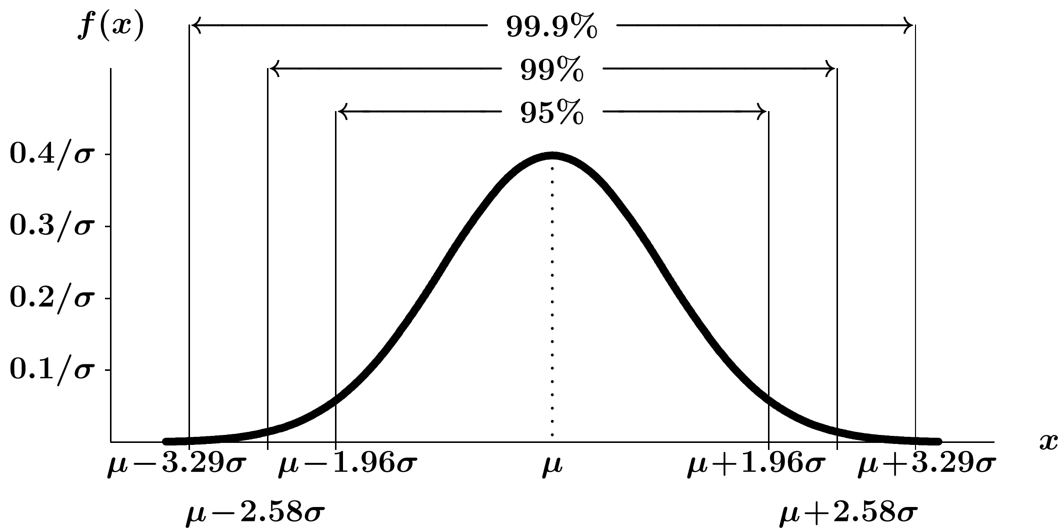
Normal Distributions

The normal distribution[5], written in PSLS and IPS as $\mathrm{N}(\mu,\sigma)$,[6] is defined by its density function/curve:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

where the mean μ and standard deviation σ are the two parameters:

- "bell-shaped" (no matter μ and σ),
- symmetrical around μ: the mean, median, and mode,
- unbounded (extends infinitely far out),
- some more probabilities (the "68 − 95 − 99.7% rule"):



interval	$[\mu-\sigma, \mu+\sigma]$	$[\mu-2\sigma, \mu+2\sigma]$	$[\mu-3\sigma, \mu+3\sigma]$
probability	68%	95%	99.7%

- probabilities not simple to calculate (more on the following pages),

[5] Also called the Gaussian distribution, after C. F. Gauss (1777-1855).
[6] It is more common to write $\mathrm{N}(\mu,\sigma^2)$ with the second parameter as the variance.

- why is the normal distribution so important?
  - mathematically very nice properties:
    - gives simple solutions to analysis-of-variance problems,
    - many "natural" distributions are well fitted by normal due to certain mathematical laws (next lecture),
  - works well, even without fitting the data perfectly,
  - historically important: before computers, it was the only distribution that enabled complex analysis!

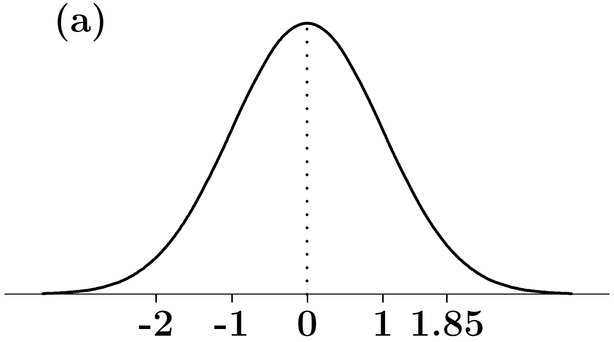
Standardized normal distribution $N(0,1)$ has $\mu = 0$ and $\sigma = 1$,

- all calculations in $N(\mu,\sigma)$ can be translated to the standard normal $N(0,1)$ (another special property),
- probabilities for $N(0,1)$ are stored in tables and computers:
  - typically $P(Z < z)$ for different z-values[7], where Z is the usual name of a random variable from $N(0,1)$,
  - tables — PSLS: Table B; S: Table 2; IPS: Table A,
  - Minitab "table": `Calc-Prob.Distribution-Normal`, choose: `Cumulative probability`; also inverse probabilities (=percentiles), at `Inverse cumulative probability`,
  - Minitab visual displays: in the menu `Graph-Probability Distribution Plot`.

[7] $P(Z < z)$ is called a cumulative probability, or (Henrik) "probability to the left".

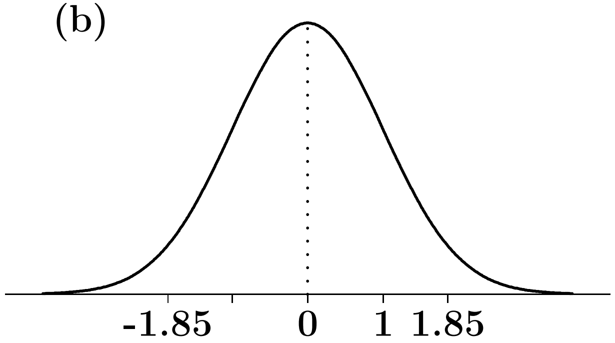
Exercise 1.109

(a)



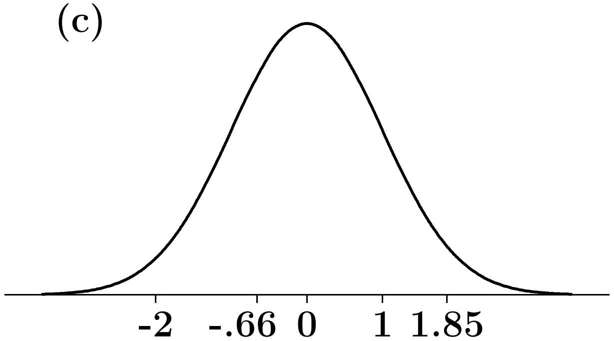
$\mathbf{P}(Z < 1.85) = 0.9678$

(b)



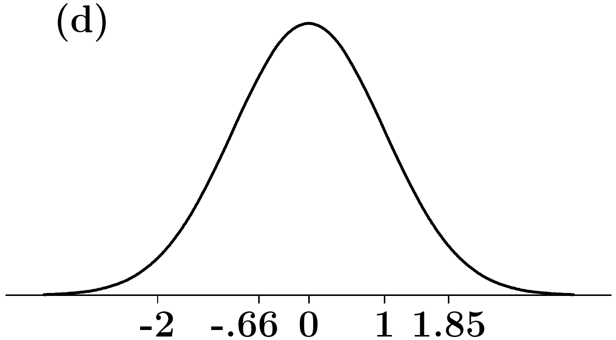
$\mathbf{P}(Z > 1.85) = 1 - 0.9678 = 0.0322$
(or computed as $\mathbf{P}(Z < -1.85)$)

(c)



$\mathbf{P}(Z > -0.66) = 1 - 0.2546 = 0.7454$
(or computed as $\mathbf{P}(Z < 0.66)$)

(d)



$$\mathbf{P}(-0.66 < Z < 1.85)$$
$$= \mathbf{P}(Z < 1.85) - \mathbf{P}(Z \leq -0.66)$$
$$= 0.9678 - 0.2546 = 0.7132$$

WORKING WITH THE NORMAL DISTRIBUTION: STANDARDIZATION

Standardization is based on the following results

(a) Any linear transformation $x \mapsto y = a + bx$ transforms a random variable $X \sim \mathrm{N}(\mu_X, \sigma_X)$ into another normal variable $Y = a + bX \sim \mathrm{N}(\mu_Y, \sigma_Y)$, where

$$\mu_Y (= \mathrm{E}Y) = a + b\,\mu_X, \qquad \text{and} \qquad \sigma_Y (= \mathrm{sd}Y) = b\,\sigma_X.$$

(b) In particular, for $X \sim \mathrm{N}(\mu_X, \sigma_X)$ we have

$$Z = (X - \mu_X)/\sigma_X \sim \mathrm{N}(0{,}1),$$

and Z is the standardized form of X.

Normal distribution calculations (PSLS Example 11.6/11.7):

The lengths of human pregnancies from conception to birth (gestation period) follow roughly a normal distribution with $\mu = 266$ and $\sigma = 16$. What proportion of babies are born after 240 days (8 months)?

We seek the probability, (where $X \sim \mathrm{N}(\mu, \sigma)$)

$$\mathrm{P}(X > 240) = \mathrm{P}\left(\frac{X - 266}{16} > \frac{240 - 266}{16}\right) = \mathrm{P}(Z > -1.625),$$

and table lookup gives $\mathrm{P}(Z > -1.63) = 1 - 0.0516 = 0.9484 = 94.8\%$; more precisely, we could have used Minitab to get: $\mathrm{P}(Z > -1.625) = \mathrm{P}(Z < 1.625) = 0.9479$. (Note: the value $z = (240 - \mu)/\sigma = -1.625$ is often called a z-score).

DETAILS FOR STANDARDIZATION EXAMPLE

A more detailed derivation of the z-score goes as follows:

$$X > 240$$

or

$$X - 266 > 240 - 266$$

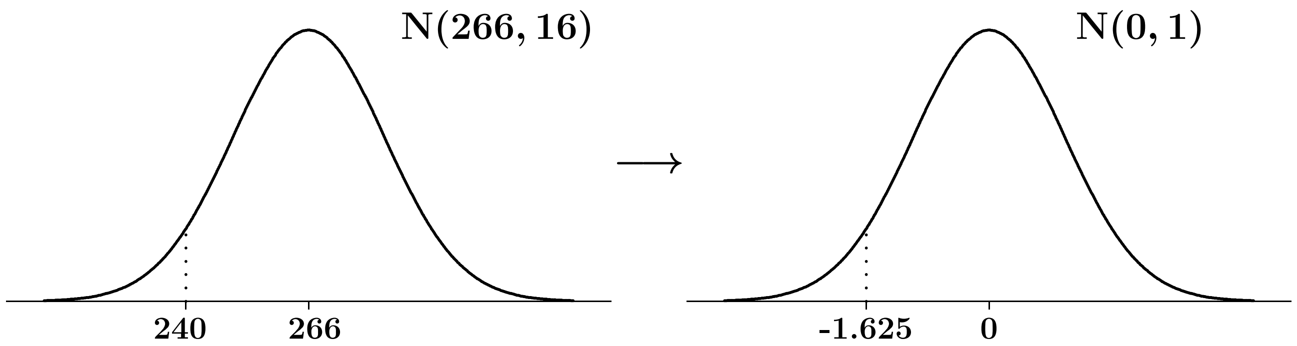
or

$$\frac{X - 266}{16} > \frac{240 - 266}{16}$$

or

$$Z > -1.625 \ (= z\text{-score})$$

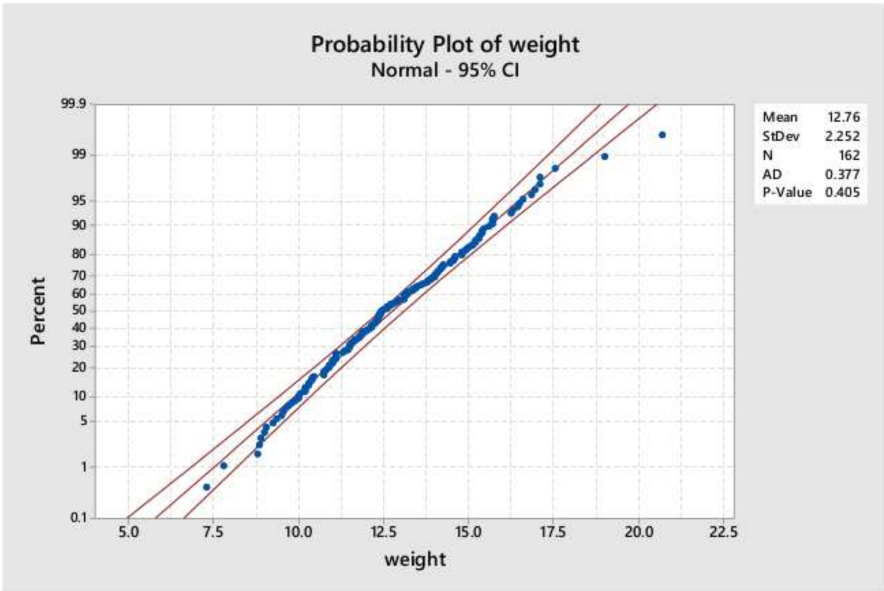
Graphically, we transfer the calculation from the left distribution to the right distribution in the figure below.



NORMAL PLOTS

Normal quantile plots (normal probability plots[8]):

- graphical assessment of normality of dataset for a single variable,
- a straight line corresponds to a normal distribution (all values of μ and σ),
- informal assessment (tests used for formal assessments),
- if the plots looks bad, always make a histogram (or stem-plot) to more clearly see the problems.



Comment: reasonably straight line, with one point at the upper end clearly off, and others just beyond the confidence bands (indicating the "expected random variation").

[8] The terms quantile plot and probability plot are used interchangeably, but usually a probability plot has percentages or z-scores on the y-axis, whereas a quantile plot usually has z-scores on the x-axis.

Introduction to Normality Tests

Main idea (more details about statistical tests later): the P-value indicates the data's agreement with a normal distribution, more precisely[9]

- low P-value: data do not appear normal; high P-value: data may very well be normal,
- definitive cut-off for low/high, such as: low $\sim P \leq 0.05$, usually not needed (the interpretation depends on the sample size, and on what the test is used for).

Implementation:

- calculation utilizes statistical software, e.g. Minitab: `Basic Stats-Normality Test` (3 tests),[10]
- many different tests exist, and they usually agree reasonably well.

Why of interest to "test for normality"...?

- if a normal distribution approximates well, we may use it for probability calculations,
- many methods for statistical inference involve assumed normal distributions
  $\Rightarrow$ use test as a validation step for the inference.

Caution: — it's one of the most misused statistical procedures...,

- the observations must be from a single distribution, not several ones pooled together (e.g., females and males),
- many methods for statistical inference do not assume the raw data values to be normally distributed (instead the "residuals"),
- mild violations of normality may not be a real problem for the validity of results.

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[9] In the statistical testing framework, the "null hypothesis H_0" is: distribution *is* normal.

[10] The Stata commands: `swilk`, `sktest` and `sfrancia` give 3 different tests; R has the command: `shapiro.test`.

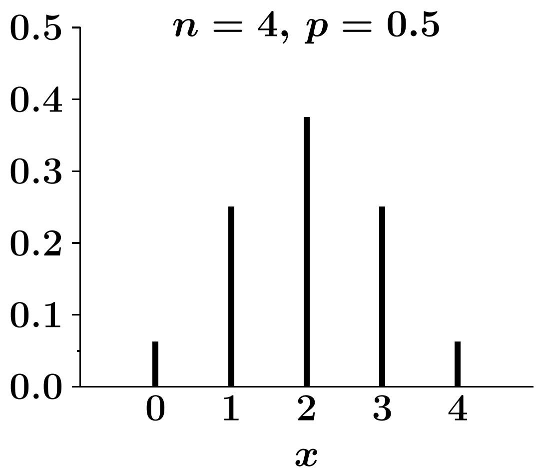
Example: Binomial Distribution

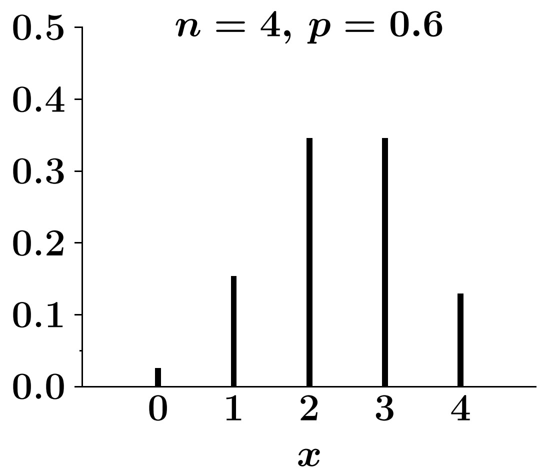
Assume we toss a coin 4 times, and that

- the coin has the same probability p to show heads (H) in every run,
- the outcomes of the 4 runs/trials are independent.

Then the total number of heads in the 4 runs follows a binomial distribution $\mathrm{B}(n, p)$ with denominator $n = 4$ and probability p. The probabilities are,

- $p(4) = \mathrm{P}(\text{“sequence HHHH”}) = p \cdot p \cdot p \cdot p = p^4$,
- $p(0) = \mathrm{P}(\text{“sequence TTTT”}) = (1 - p)^4$,
- $p(3) = \mathrm{P}(\text{“sequence HHHT, HHTH, HTHH or THHH”}) = 4p^3(1 - p)$,
- $p(1) = 4p(1 - p)^3$,
- $p(2) = 6p^2(1 - p)^2$.





BINOMIAL SETTING

A binomial setting involves the following assumptions,

- a fixed number n of observations (often "trials"),
- the n observations are all independent,
- each observation takes one of two possible values (categories) — "success" (1) or "failure" (0),
- the probability of "success" is the same, p, for all observations.

$\Rightarrow$ a binomial distribution $\mathrm{B}(n, p)$ for the number of successes (out of the total n).

Typical examples: outcomes such as germination, survival, presence/absence of certain phenomena (especially disease), answer yes/no to questions.

Sampling of n elements from a finite population (N elements) with two categories of elements (of which proportion p is of first category),

- *(rare)*: with replacement $\Rightarrow$ binomial setting (n, p),
- *(in practice)*: without replacement $\Rightarrow$ approximate binomial setting when $n \ll N$; rule of thumb for use of approximation is that $N \geq 20n$ (or, $n/N \leq 5\%$).

Sampling without replacement from a finite population is modelled (exactly) by a hypergeometric distribution (not covered in our textbooks).

Exercises 5.32 and 5.33

Using a binomial distribution.

Exercise 5.32:

(a) A binomial distribution $B(50, p)$, where $p =$ probability of girl, seems reasonable, under one assumption about the data collected. (which? — no twins)

(b) Clearly not a binomial distribution. (why not? — no fixed n)

(c) It's probably not reasonable to assume a binomial distribution $B(50, p)$. (why not? — dependence within a couple, not same probability for women and men)

Exercise 5.33:

(a) A binomial distribution $B(50, p)$, where $p =$ probability of passing the exam. One should consider which population the students are intended to represent, and also whether they were taught together or individually.

(b) Not a binomial distribution $B(10, p)$. (why not? — because of the instruction between questions, p is not constant)

(c) Not a binomial distribution $B(10, p)$. (why not? — p not constant, due to the different temperatures)

Binomial Distribution II

Binomial distribution (n, p) (or $\mathrm{B}(n, p)$ or $\mathrm{Bin}(n, p)$), for a binomial setting:

- $n =$ number of trials/observations (denominator),
- $p =$ probability parameter, $0 \leq p \leq 1$,

has the following properties,

- the probability function $p(x)$ is given by
$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, \ldots, n,$$
where the binomial coefficient $\binom{n}{x} = \frac{n!}{x!\,(n-x)!}$ is the number of ways to select a subset of x elements from a set of n elements ("n choose x"), and $n!$ is the factorial[11],
- mean $\mu = np$, and standard deviation $\sigma = \sqrt{np(1-p)}$.

Probabilities in $\mathrm{B}(n, p)$ can be obtained by

- calculation by hand/calculator,
- tables: PSLS: no table; S: Table 1; IPS: Table C — for selected values of (n, p),
- Minitab[12] "table" menu: `Calc-Probability Distribution-Binomial`,
- approximation formulas (next lecture).

[11] Factorial function: $n! = 1 \cdot 2 \cdots (n-1) \cdot n$, for integers $n \geq 1$, and $0! = 1! = 1$.

[12] The Stata function: `binomial`(n, k, p) gives $\mathrm{P}(X \leq k)$ for $X \sim \mathrm{B}(n, p)$; in R, the function is: `pbinom`(k, n, p).

Statistical Inference

– to draw conclusions from data about some wider population (for which the data are representative),

– relies always on assumptions of statistical models,

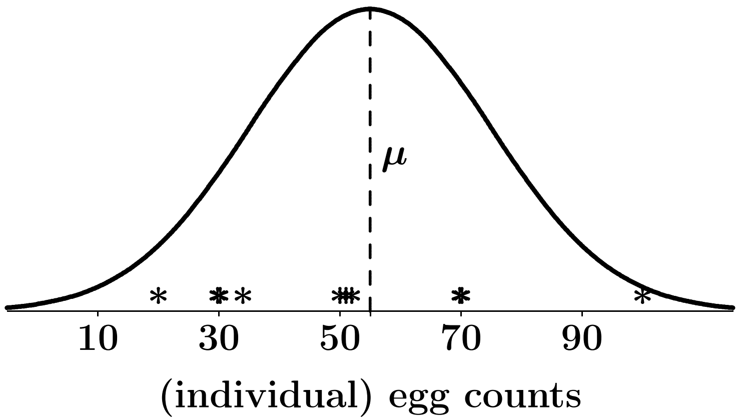
– methods developed to supplement, not substitute, our common sense.

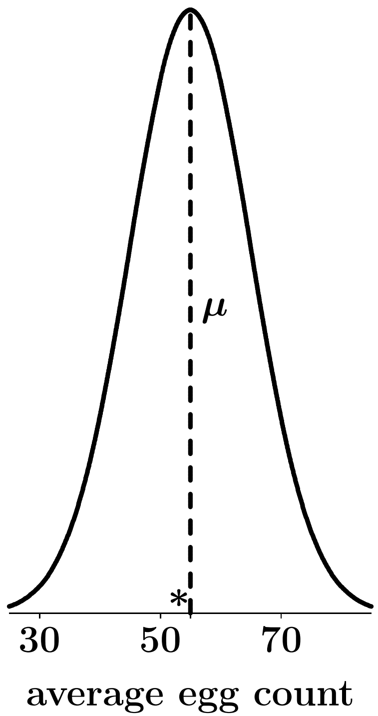
Outline of statistical analysis:

- Data description (Descriptive statistics).
- Statistical models: we formulate statistical models involving theoretical probability distributions and their parameters (e.g., μ):
  - fixed, unknown numbers describing a population,
  - the primary focus of statistical inference.
- Estimation: our information about the parameters from the observed data is summarized in statistics or estimates:
  - calculated from the data to give possible parameter values, e.g. the sample mean $\bar{X}$,
  - standard notation: parameters with a "hat", e.g. $\hat{\mu} = \bar{X}$,
  - aim of statistical methodology: obtain estimates as close to true values as possible (though *rarely* (never) equal to true values),
  - any estimate should be accompanied by a measure of its uncertainty: e.g. its standard deviation (often called standard error, SE) or a confidence interval.

Example: Parasite Burdens in Lithuania

Observed (infected pasture group) and average egg count, with overlaid hypothetical population (normal) distributions:





- population: calves on infected pasture, under similar conditions as in spring-summer in Lithuania,
- model: sample of size 10 from $N(\mu, \sigma)$,
- parameters: μ and σ = population mean and standard deviation of egg counts after 10 weeks on pasture,
- estimates:

$$\hat{\mu} = \bar{X} = 51.2 \quad \text{(sample average)},$$
$$\hat{\sigma} = s = 24.0 \quad \text{(sample standard deviation)}.$$

Example: Diagnostic Testing

Assume a new laboratory diagnostic test under development for detection of a disease; e.g. ISA (infectious salmon anemia).

Two essential characteristics of the test:

- sensitivity = proportion of true positive animals detected by the test,[13]
- specificity = proportion of true negative animals declared negative by the test.[13]

Interpretations (sensitivity):

- population: "true positive animals" — must be defined by some other criterion/test (maybe a "gold standard"),
- model: each infected animal has the same probability p of testing positive by the test $\Rightarrow$ Binomial distribution $\mathrm{B}(n, p)$;
  - * do population characteristics such as age, sex, severity of disease etc., affect the probability of testing positive?
- parameter: p (sensitivity),
- estimate: $\hat{p}$ = proportion of positives in sample.

In practice, sensitivity and specificity calculations are limited to well-defined and often fairly narrow populations.

[13] We already (Lecture 3) encountered sensitivity (Se) and specificity (Sp); recall that Se = 100% $\Rightarrow$ no false negatives, and Sp = 100% $\Rightarrow$ no false positives.

Distribution of a Statistic

Let our data be (random variables) $X_1, \ldots, X_n$ (e.g., the parasite egg counts).

More formally about estimates:

- estimates (in fact, all statistics) are functions of the data:

$$X_1, \ldots, X_n \mapsto f(X_1, \ldots, X_n),$$

  for example, the average $\bar{X} = (X_1 + \ldots + X_n)/n$,

- estimates are random variables and have distributions;
  — intuitively, if the X's are random, so is the average,

- the distribution of estimates depends on the unknown parameter(s) of the model;
  — intuitively, if the distribution of the X's does, the same must be true for statistics computed from them.

But . . . where does the variation come from?

- sometimes sampling variability, when randomly selecting a sample from a population (because different members of the population get selected),
- sometimes physical variation, like measurement error, laboratory variation etc.,
- sometimes biological variation, when "homogeneous" experimental units (e.g. animals) react differently upon experimental conditions.

SUMMARY NOTES

The normal distribution $\mathrm{N}(\mu, \sigma)$:

- parameters: μ (mean) and σ (standard deviation),
- symmetry, "bell-shape", $68 - 99 - 99.7$ rule,
- standard normal ($\mathrm{N}(0, 1)$, Z-distribution), and methods for getting probabilities/percentiles,
- standardization, z-score,
- normal quantile/probability plot, normality test.

The binomial distribution $\mathrm{B}(n, p)$:

- parameters n (number of trials) and p (probability),
- binomial setting, approximation for simple random sampling: 5% rule,
- probability function, mean, standard deviation.

The Poisson distribution with mean λ for counts of events[14]

- probability function: $p(x) = \mathrm{e}^{-\lambda}\lambda^x/x!$, for $x = 0, 1, 2 \ldots$

Key words and concepts in statistical inference:

- parameter, estimate, population,
- distribution of estimate/statistic,
- statistical model (a formal set of assumptions needed to make the inference valid).

[14] Counts with no natural upper bound and sample space $S = \{0, 1, 2, \ldots\}$; typical examples: traffic accidents, cases of a non-infectious disease, bacteria colonies on a plate, plants in an area.