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Practical Information

Lecture contents:

- brief follow-up on sample size calc. (8L—17),
- 2-sample proportions revisited (7L—15/16), incl. z-tests,
- 2-way (contingency) tables and chi-square tests[1] (also Simpson's paradox and 2-way table calculations),[2]
- include in course: the distinction between the 2 models,
- two additional topics:
  - multinomial distribution (*in* course curriculum),
  - Fisher's exact test (*not in* course curriculum).

Schedule news:

- home assignment II and midterm both returned next week; solutions will be posted when the marking is done,
- home assignment III to be posted on November 1st.

[1] PSLS 3e: Chapter 23; S: Section 11.2; IPS 7e: Sections 9.1-2.

[2] PSLS 3e: Chapter 5; IPS 7e: Section 2.5; not covered in S (note that S discusses Simpson's paradox for quantitative data only).

Data Example: Avadex for Mice

A clinical trial was conducted to assess a possible carcinogenic effect of Avadex (a fungicide).

- Data: control and treatment (Avadex in feed) groups of mice; number of mice with lung tumors recorded:

Outcome	Avadex group	control group	Total
tumors	4	5	9
no tumors	12	74	86
Total	16	79	95

- Model: $X \sim \mathrm{B}(16, p_1)$ and $Y \sim \mathrm{B}(79, p_2)$, where X, Y are no. of mice with tumors in Avadex/control groups,
- Estimation:

$$\hat{p}_1 = 4/16 = 0.250, \quad \mathrm{SE}_{\hat{p}_1} = 0.108,$$
$$\hat{p}_2 = 5/79 = 0.063, \quad \mathrm{SE}_{\hat{p}_2} = 0.027,$$
$$\hat{p}_1 - \hat{p}_2 = 0.187, \quad \mathrm{SE}_{\hat{p}_1 - \hat{p}_2} = 0.112,$$

- Confidence intervals: (95%, plus four method)

  $p_1 : 0.30 \pm 0.20$, $p_2 : 0.084 \pm 0.060$, $p_1 - p_2 : 0.204 \pm 0.215$.

- Hypotheses: H_0: $p_1 = p_2$ $(= p)$ vs. H_a: $p_1 \neq p_2$; estimate common p under H_0: $\hat{p} = 9/95 = 0.095$, and $\mathrm{SE}_{D_p} = \sqrt{\hat{p}(1-\hat{p})((1/16) + (1/79))} = 0.080$,
- Test: (approximate, based on normal distribution) $z = (\hat{p}_1 - \hat{p}_2)/\mathrm{SE}_{D_p}$, $z_{\mathrm{obs}} = 0.187/0.080 = 2.326$, which gives P-value $= 2 \cdot \mathrm{P}(Z > 2.326) = 0.020$.

Data Example: Health Habits of Students

A survey[3] obtained information on the levels of physical activity and consumptions of fruit — is there a link between these or they are independent?

Data: responses obtained for 1184 college students:

Fruit consumption	Physical activity low	moderate	vigorous	Total
low	69	206	294	569
medium	25	126	170	321
high	14	111	169	294
Total	108	443	633	1184

2 response variables, because none of the variables are fixed (known) in advance.

Descriptive statistics: (*for response variables*)

- Marginal distributions — looking at each variable separately: fruit consumption: 569/1184=48% low, 321/1184=27% medium, 294/1184=25% high; physical activity: 108/1184=9% low, 443/1184=37% moderate, 633/1184=53% vigorous,
- Conditional distributions — looking at one variable when the other is fixed: e.g. fruit consumption in low physical activity group: 69/108=64% low, 25/108=23% medium, 14/108=13% high.

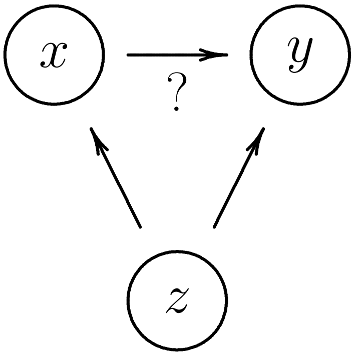
[3] Data from Seo D-C et al. (2007), *Amer. J. College Health* **56**, 187-197; also Example 9.8 of IPS 7e.

SIMPSON'S PARADOX

= an extreme affect of ignoring a lurking variable,

– at closer look, no paradox at all.

Example: Airline punctuality (IPS7e Suppl. Exercise 9.19):



x = airline (Alaska Airlines/America West)
y = on time (yes/no)
z = airport (Los Angeles/Phoenix)

Summary 2-way tables:

	All airports		Los Angeles		Phoenix	
On time/airline	AA	AW	AA	AW	AA	AW
on time	718	5534	497	694	221	4840
delayed	74	532	62	117	12	415
sum	792	6066	559	811	233	5255
prop. delayed	0.093	0.088	0.111	0.144	0.052	0.079

Comments and conclusions:

- Simpson's paradox:
  - overall, airline AW better (less delays) than AA,
  - in both airports, airline AA better than AW,
- explanation: airport Los Angeles has more flights delayed, and AA has more flights at this airport,
- conclusion: "paradox" may happen whenever both effects $z \to x$ and $z \to y$ are strong.

Data Example: Music and Wine Purchase

Experimental study on music's impact on wine purchase (no. of bottles sold of categories French, Italian and Other) in a supermarket[4] under different music conditions (None, French accordion music, Italian string music).

Data: 243 bottles sold categorized by wine and music types:

# bottles	Music			
Wine	None	French	Italian	Total
French	30	39	30	99
Italian	11	1	19	31
Other	43	35	35	113
Total	84	75	84	243

1 response variable — the type of wine purchased,
1 explanatory variable — the type of music played (controlled by the store) $\sim$ 3 separate time periods and therefore independent samples.

Descriptive statistics:

- Conditional distributions — proportions of wine sold for the three samples: e.g., Italien wine $\sim 19/84 = 23\%$ for Italian music, but only $1/75 = 1\%$ for French music,
- Marginal wine type distribution — pooled across music type: e.g., Italian wine $\sim 31/243 = 13\%$ of bottles sold.

[4] Study carried out in Northern Ireland in 1990s; Ryan et al. (1998), Proc. Nutrition Soc. **57**, 169A; also Example 9.8 in IPS 6e.

Multinomial Distribution

Example: wine purchase when no music is played:

type of wine	French	Italian	Other	Total
count	30	11	43	84
symbol N_i	N_1	N_2	N_3	n
rel. frequency	0.357	0.131	0.519	1
symbol p_i	p_1	p_2	p_3	1

Multinomial Distribution:

$$(N_1, N_2, \ldots, N_q) \sim \text{multinomial}(n; p_1, p_2, \ldots, p_q)$$

where n is the total number of observations, q is the number of classes, and p_i is the (population) probability of each class (so that $p_1 + p_2 + \ldots + p_q = 1$), if

- mathematical definition:

$$\mathrm{P}(N_1 = n_1, \ldots, N_q = n_q) = \binom{n}{n_1 \ldots n_q} p_1^{n_1} \cdots p_q^{n_q},$$

  where *multinomial coefficient* $\binom{n}{n_1 \ldots n_q} = \frac{n!}{n_1! \cdots n_q!}$,

- conceptual definition ("multinomial setting"):
  - ∗ n trials; in each, one of q categories is observed,
  - ∗ *independent* trials, with *same probabilities* of the categories across all trials.

Note: As Multinomial$(n; p_1, p_2) = \mathrm{B}(n, p_1)$, the multinomial distribution extends the binomial to > 2 categories.

2-Way Tables: Notation

$I \times J$ table:
n observations grouped (cross-classified) according to two criteria A and B with I and J levels, respectively:

Counts		$j \sim$ criterion B							
		1	2	$\dots$	j	$\dots$	$J-1$	J	sum
	1	N_{11}	N_{12}	$\dots$	N_{1j}	$\dots$	$N_{1,J-1}$	N_{1J}	$N_{1\cdot}$
	2	N_{21}	N_{22}	$\dots$	N_{2j}	$\dots$	$N_{2,J-1}$	N_{2J}	$N_{2\cdot}$
$i \sim$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
crit.	i	N_{i1}	N_{i2}	$\dots$	N_{ij}	$\dots$	$N_{i,J-1}$	N_{iJ}	$N_{i\cdot}$
A	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$
	$I-1$	$N_{I-1,1}$	$N_{I-1,2}$	$\dots$	$N_{I-1,j}$	$\dots$	$N_{I-1,J-1}$	$N_{I-1,J}$	$N_{I-1,\cdot}$
	I	N_{I1}	N_{I2}	$\dots$	N_{Ij}	$\dots$	$N_{I,J-1}$	N_{IJ}	$N_{I\cdot}$
	sum	$N_{\cdot 1}$	$N_{\cdot 2}$	$\dots$	$N_{\cdot j}$	$\dots$	$N_{\cdot,J-1}$	$N_{\cdot J}$	n

Notes:

- always: $i \sim$ rows, and $I =$ no. of rows,
  $j \sim$ columns, and $J =$ no. of columns,
- each (i, j) combination corresponds to a table cell,
- the $N_{i\cdot}$'s are row totals, and the $N_{\cdot j}$'s are column totals,
- textbook chapters are "notation-less", but double subscript notation will be used later on (for ANOVA).

2-Way Tables: Models and Estimation

Model I: Independent multinomials[5] over columns (or rows):

$$\begin{aligned}(N_{11},\ldots,N_{I1}) &\sim \text{ multinomial } (N_{.1}; p_{11},\ldots,p_{I1}),\\ &\vdots\\ (N_{1J},\ldots,N_{IJ}) &\sim \text{ multinomial } (N_{.J}; p_{1J},\ldots,p_{IJ}),\end{aligned}$$

where p_{ij} = probability of group i in jth column, and all probability column sums equal 1 (e.g., $p_{11} + \ldots p_{I1} = 1$).

- Examples: Wine purchase, Avadex in mice,
- Assumptions: multinomial setting in each column, and independence between columns,
- Estimation: $\hat{p}_{ij} = N_{ij}/N_{.j}$ — sample proportions within each column,
- Interpretation: one response variable (rows), one explanatory variable (columns).

Model II: a Single multinomial[6] on IJ classes:

$$(N_{11},\ldots,N_{ij},\ldots,N_{IJ}) \sim \text{ multinomial } (n; p_{11},\ldots,p_{ij},\ldots,p_{IJ}),$$

where p_{ij} = probability of group (cell) (i,j), and all probabilities sum to 1 (i.e., $p_{11} + p_{12} + \ldots + p_{1J} + p_{21} + \ldots + p_{2J} + \ldots + p_{IJ} = 1$),

- Example: Health habits,
- Assumptions: multinomial setting for table (IJ cells),
- Estimation: $\hat{p}_{ij} = N_{ij}/n$ — table sample proportions,
- Interpretation: 2 response variables (rows and columns).

[5] IPS: model for comparing several populations or independent SRSs.

[6] IPS: model for examining independence or for a single SRS.

2-Way Tables: Hypotheses

Model I: Independent multinomials over columns:

- Hypothesis H_0: homogeneity among columns (same distribution in all columns):

$$\mathrm{H}_0 : p_{ij} = p_{i\cdot} \text{ for all } j, \quad \text{and } \mathrm{H}_a : \text{ not } \mathrm{H}_0,$$

  and H_0 corresponds to using the marginal distribution across columns (row totals),

- Estimation under H_0: $\hat{p}_{i\cdot} = N_{i\cdot}/n$,
- Expected value of cell (i,j) under H_0:
  e_{ij} = row total $\times$ column total / overall total.

Model II: a Single multinomial on IJ classes:

- Hypothesis H_0: independence between row and column classification:

$$\mathrm{H}_0 : p_{ij} = p_{i\cdot}\, p_{\cdot j} \text{ for all } i \text{ and } j, \quad \text{and } \mathrm{H}_a : \text{ not } \mathrm{H}_0,$$

  and H_0 corresponds to using the marginal distribution across both rows and columns,

- Interpretation:

$$\begin{aligned} p_{ij} &= \mathrm{P}(\text{row}=i \text{ and column}=j) \\ &= (\textit{independence})\ \mathrm{P}(\text{row}=i)\,\mathrm{P}(\text{column}=j) = p_{i\cdot}\, p_{\cdot j}, \end{aligned}$$

- Estimation under H_0: $\hat{p}_{i\cdot} = N_{i\cdot}/n$, and $\hat{p}_{\cdot j} = N_{\cdot j}/n$,
- Expected value of cell (i,j) under H_0:
  e_{ij} = row total $\times$ column total / overall total.

2-Way Tables: Test

Result: In *both* of the models I and II, we test H_0 (homogeneity or independence) by the (Pearson chi-square) statistic,

$$X^2 = \sum_{i,j} \frac{(N_{ij} - e_{ij})^2}{e_{ij}} = \sum_{i,j} \frac{(\text{observed}_{ij} - \text{expected}_{ij})^2}{\text{expected}_{ij}}$$
$$\sim \chi^2 \text{ distribution(df) under } H_0,$$

where df $= (I-1)(J-1)$.[7]

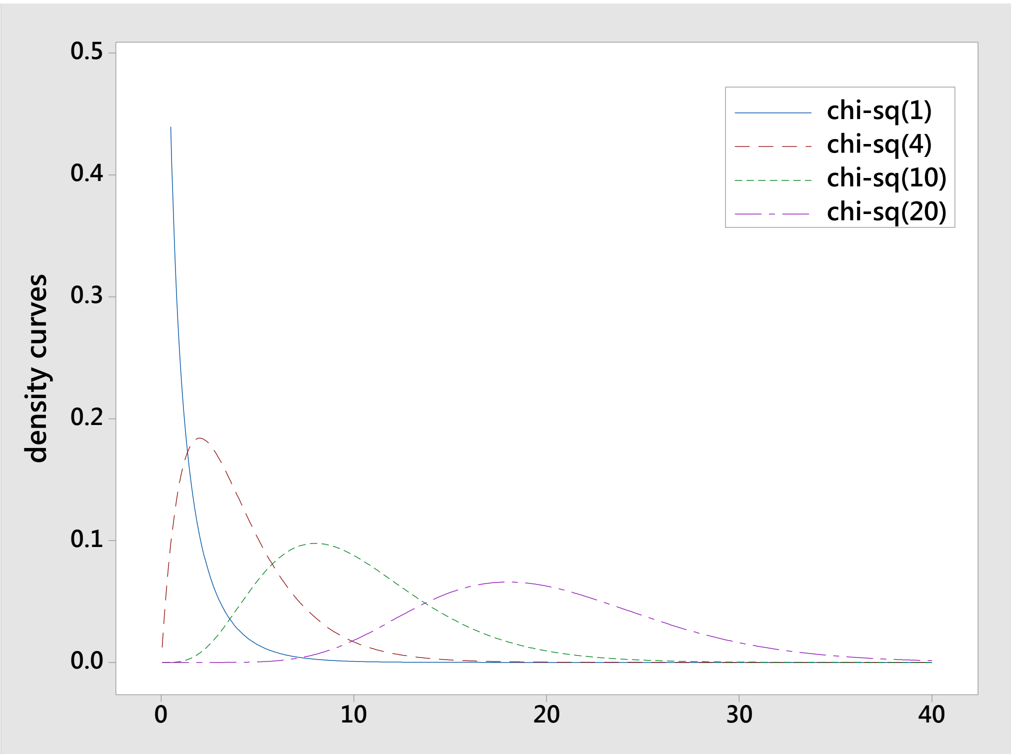
Notes:

- *same* statistic, but *different* models and hypotheses $\Rightarrow$ different interpretations!
- one-sided test: large values are critical, and $P = \mathrm{P}(\chi^2(\text{df}) > X^2_{\text{obs}})$,
- distribution of X^2 under H_0 is approximate and best for large n; guidelines for use:
  - $e_{ij} > 1$ in all cells (i, j), and
  - $e_{ij} > 5$ in at least 80% of cells (i, j),
- other test for H_0 exist and may be better…

[7] Technical note: the degrees of freedom can in both models be calculated as number of free parameters in model minus number of free parameters under H_0.

χ^2 Distributions

- "chi-square" distribution,
- a new distribution — to be used for test statistics in tables of categorical data (*not for modelling*):
  — reference distribution for X^2 (or χ^2) statistics,
- a single parameter df (df $= 1, 2, 3, \ldots$), the degrees of freedom (determined by the statistical design),
- denoted χ^2(df) to indicate degrees of freedom,
- distribution on $(0, \infty)$ — only positive values,
- mean $=$ df, standard deviation $= \sqrt{2\text{df}}$,
- tail probabilities: software, or tables: PSLS Table D; S Table 5; IPS Table F.



Music and Wine Purchase — Analysis

- Model: three independent multinomial distributions on 3 classes,
- Hypothesis H_0: same distribution of wine purchases for all types of music,
- Test: see table below for computation of X^2:
  - $\text{df} = (3-1) \cdot (3-1) = 4$,
  - $X^2_{\text{obs}} = 18.28$, $P = \text{P}(\chi^2(4) > 18.28) = 0.0011$,
- Conclusion: very clear evidence against $H_0 \Rightarrow$ wine purchase depends on music played,
- Presentation and estimation: observed proportions separately for the three samples.

Table of computations for X^2:

count (e_{ij})	Music			
Wine	None	French	Italian	Total
French	30 (34.2)	39 (30.6)	30 (34.2)	99
Italian	11 (10.7)	**1** (**9.6**)	**19** (**10.7**)	31
Other	43 (39.1)	35 (34.9)	35 (39.1)	113
Total	84	75	84	243

$$X^2 = \frac{(30-34.2)^2}{34.2} + \frac{(39-30.6)^2}{30.6} + \ldots + \frac{(35-39.1)^2}{39.1} = 18.28.$$

Health Habits — Analysis

- Model: a single multinomial distribution on 9 classes,
- Hypothesis H_0: independence between levels of fruit consumption and exercise,
- Test: see table below for computation of X^2:
  - ∗ $\text{df} = (3-1) \cdot (3-1) = 4$,
  - ∗ $X^2_{\text{obs}} = 14.15$, $P = \text{P}(\chi^2(4) > 14.15) = 0.007$,
- Conclusion: very clear evidence against $H_0 \Rightarrow$ dependence between fruit consumption and exercise level,
- Presentation and estimation: conditional distributions for fruit consumption given physical activity, or conversely; the main discrepancies are in cells (low,low) and (high,low).

Table of computations for X^2:

count (e_{ij})	Physical activity			
Fruit	low	moderate	vigorous	Total
low	**69** (**51.9**)	206 (212.9)	294 (304.2)	569
medium	25 (29.3)	126 (120.1)	170 (171.6)	321
high	**14** (**26.8**)	111 (110.0)	169 (157.2)	294
Total	108	443	633	1184

$$X^2 = \frac{(69-51.9)^2}{51.9} + \frac{(206-212.9)^2}{212.9} + \ldots + \frac{(169-157.2)^2}{157.2} = 14.15.$$

2×2 Tables

2×2 tables = special case of 2-way tables:

- simplest case, and very common in practice,
- special relations for chi-square statistic:
  - $X^2 = z^2$, where z is statistic for comparing two binomial distributions,
    $\Rightarrow$ methods equivalent (lead to same P-value), and guideline for use of X^2 applies to z as well!
  - easier computational formula for X^2:

$$X^2 = \frac{(N_{11}N_{22} - N_{12}N_{21})^2}{N_{1.}N_{2.}N_{.1}N_{.2}} \times n$$

- an almost endless selection of methods / procedures:
  - alternative tests for same hypothesis:
    - Fisher's "exact" test (next slide),
    - continuity-correction for X^2 (better approximated by χ^2-distribution); use Fisher's test instead,
  - other measures for comparison of probabilities than differences: relative risk and odds-ratio (epi course),
  - tests for other hypotheses (e.g. McNemar's test) . . . ,
  - other statistics (e.g. kappa values) . . .
- simple advice: use your common sense and use only procedures that you understand (the rationale and assumptions behind).

Fisher's Exact Test

Fisher's exact test for 2 independent binomial distributions:

- test of null hypothesis H_0: $p_1 = p_2$ against one- or two-sided alternative H_a,
- test "statistic" = observed table (one cell of table),
- idea: compare observed table with other tables that have the *same margins* (row and column sums),
- technical: under H_0, tables with same margins $\sim$ hypergeometric distribution $\Rightarrow$ table probabilities computable,
- P-value for one-sided H_a = sum of table probabilities for tables more indicative for H_a than for H_0,
- P-value for two-sided H_a = twice the smallest one-sided P, or the sum of table probabilities less than for observed table (most commonly used: Minitab/Stata/R).

Why / When use Fisher's test?

+ works also if χ^2-approximation not good
  $\Rightarrow$ recommended test if χ^2-guidelines violated,
+ allows one-sided alternative hypothesis (as z-tests),

− requires software, and for large samples computing time may be very long,

∗ also for multinomial model and test of independence,

∗ version of test also for larger tables than 2×2.

Avadex Example: Fisher's Exact Test

Data table (with expected values) and general notation:

Outcome	Avadex	control	Total		Avadex	control	Total
tumors	4 (1.5*)	5 (7.5)	9		a	b	K
no tumors	12 (14.5)	74 (74.5)	86		c	d	$N-K$
Total	16	79	95		n	$N-n$	N

* expected value $< 5 \Rightarrow$ conditions for X^2-test violated

Idea: under H_0, we consider all table margins as fixed and focus only on the distribution of the $K = 9$ tumor cases among the two samples,

- similar to sampling of n elements from a finite population of size N of which K are "cases" $\rightarrow$ a hypergeometric distribution (N, K, n, a) for the number of cases a in the sample.

	Cell counts				Prob.	One/two-sided tail prob.		
Scenario	a	b	c	d	under H_0	one- ($>$)	one- ($<$)	two- ($\neq$)
1	0	9	16	70	.175		.175	
2	1	8	15	71	.335		.335	
3	2	7	14	72	.296		.296	
4	3	6	13	73	.132		.132	
5	**4**	**5**	**12**	**74**	**.035**	.035	.035	.035
6	5	4	11	75	.006	.006		.006
7	6	3	10	76	.001	.001		.006
8	7	2	9	77	.000	.000		.000
9	8	1	8	78	.000	.000		.000
10	9	0	7	79	.000	.000		.000
P-value = sum of tail prob.						.041	.994	.041

Note: two-sided P-value computed by adding up prob. for tables with prob. $\leq$ observed table; here, it equals one of the one-sided P-values.

Summary Notes

Key words and concepts:

- two-way table (of counts), 2×2-table (and larger),
- marginal and conditional distributions in two-way tables, Simpsons paradox,
- multinomial distribution for counts in > 2 categories,
- 2 models/hypotheses for two-way tables of counts:
  - ∗ independent multinomial distributions (for comparing several populations/samples),
    with hypothesis of *homogeneity*,
  - ∗ single multinomial distribution for entire table (for single population/sample),
    with hypothesis of *independence*,
- X^2-test statistic:
  - ∗ computation from observed and expected counts,
  - ∗ reference χ^2-distribution, and its degrees of freedom,
  - ∗ guideline for use of X^2-test (and its χ^2 reference distribution),
  - ∗ relationship with 2-sample z-test (for 2×2-table),
- Fisher's exact test for sparse tables.